Understanding Pure Mathematics

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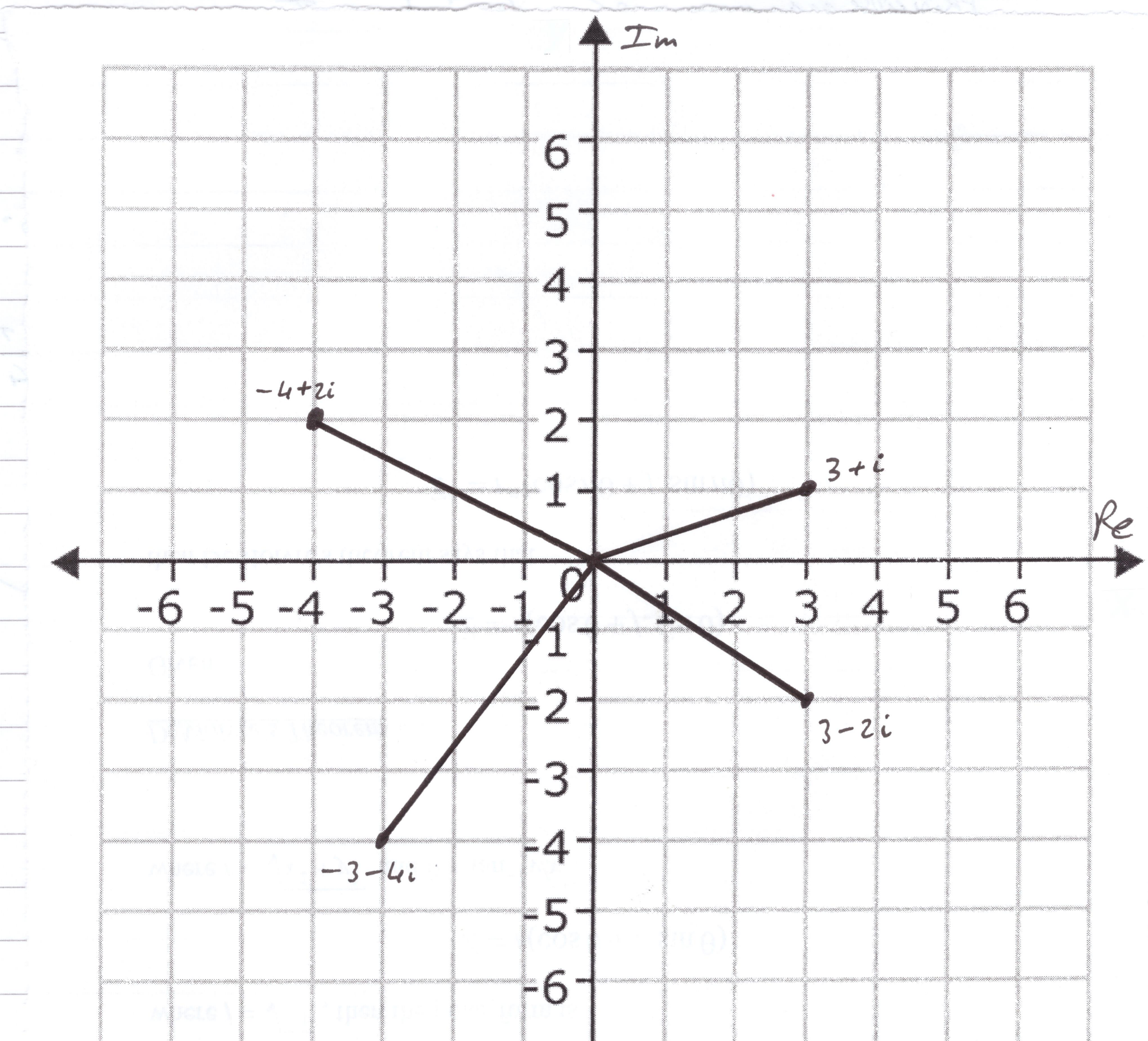
Chapter 18 : Algebra II

Exercise 18 E

(1) $\overrightarrow{OA} = 3 + 2i$; $\overrightarrow{OB} = -1 + 3i$; $\overrightarrow{OC} = -3 - 4i$

$\overrightarrow{OD} = 4 - i$

(2)



③ $\vec{OA} = 5(\cos 30^\circ + i \sin 30^\circ) = 5\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$

$\vec{OB} = 4(\cos 135^\circ + i.\sin 135^\circ) = 4\left(\cos\frac{3\pi}{4} + i.\sin\frac{3\pi}{4}\right)$

$\vec{OC} = 3(\cos(-60^\circ) + i.\sin(-60^\circ)) = 3\left(\cos\frac{\pi}{3} - i.\sin\frac{\pi}{3}\right)$

$\vec{OD} = 3(\cos(-120^\circ) + i.\sin(-120^\circ)) = 3\left(\cos\frac{2\pi}{3} - i.\sin\frac{2\pi}{3}\right)$

④ ⓐ modulus $= \sqrt{1^2+1^2} = \sqrt{2}$

Principle arg $\theta = \tan^{-1}\frac{1}{1} = \frac{\pi}{4}$

ⓑ modulus $= \sqrt{2^2+(-2)^2} = \sqrt{8}$

arg $\theta = \tan^{-1}\left(\frac{2}{-2}\right) = -45^\circ$

So Principle arg $\theta = -45^\circ + 180^\circ = 135^\circ = 3\pi/4$

ⓒ modulus $= \sqrt{(-2)^2+0^2} = 2$

arg $\theta = \tan^{-1}\left(\frac{0}{-2}\right) = 0$. So Principle arg $\theta = 0 + 180^\circ = 180^\circ$
$= \pi$

ⓓ modulus $= \sqrt{0^2+(-3)^2} = 3$; Principle arg $\theta = \tan^{-1}\left(\frac{-3}{0}\right) = -90^\circ$
$= -\pi/2$

ⓔ modulus $= \sqrt{(-2)^2+(-2)^2} = \sqrt{8}$

arg $\theta = \tan^{-1}\left(\frac{-2}{-2}\right) = 45^\circ$. So Principal arg $\theta = 45^\circ - 180^\circ$
$= -135^\circ$
$= -\frac{3\pi}{}$

(5) (a) $z = 5i$, $\therefore |z| = \sqrt{0^2 + 5^2} = 5$

Principle arg $\theta = \tan^{-1}\left(\frac{5}{0}\right) = 90^\circ = \frac{\pi}{2}$

(b) $z = 7$, $\therefore |z| = \sqrt{7^2 + 0^2} = 7$

Principle arg $\theta = \tan^{-1} \frac{0}{7} = 0$

(c) $z = -2i$, $\therefore |z| = \sqrt{0^2 + (-2)^2} = 2$

Principle arg $\theta = \tan^{-1}\left(\frac{-2}{0}\right) = -\frac{\pi}{2}$

(d) $z = -3$, so $|z| = \sqrt{(-3)^2 + 0^2} = 3$

arg $\theta = \tan^{-1}\left(\frac{0}{-3}\right) = 0$, so principle arg $\theta = 0 + \pi = \pi$

(e) $z = 1 + \sqrt{3}i$, so $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$

Principle arg $\theta = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3}$

(f) $z = 5\sqrt{3} - 5i$, so $|z| = \sqrt{(5\sqrt{3})^2 + (-5)^2} = 10$

arg $\theta = \tan^{-1} \frac{-5}{5\sqrt{3}} = -\frac{\pi}{6}$, so This is Principle argument

(g) $z = 3-4i$, so $|z| = \sqrt{3^2 + (-4)^2} = 5$

Principal arg $\theta = \tan^{-1} \frac{-4}{3} = -53.13^\circ = -0.93^c$

(h) $z = -5 + 12i$, so $|z| = \sqrt{(-5)^2 + 12^2} = 13$

$\arg \theta = \tan^{-1} \frac{12}{-5} = -67.38^\circ = -1.18^c$

So Principal $\arg \theta = \pi - 1.18^c = 1.97^c$

(6) $z_1 = 5\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = 5i$

$z_2 = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i \cdot \sin\left(-\frac{\pi}{4}\right)\right) = 4\sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$

$= 4 - 4i$

$z_3 = 4\left(\cos \pi + i \cdot \sin \pi\right) = -4$

$z_4 = 12\left(\cos \frac{2\pi}{3} + i \cdot \sin \frac{2\pi}{3}\right) = -6 + 6\sqrt{3} i$

$z_5 = 4\left(\cos\left(-\frac{\pi}{3}\right) + i \cdot \sin\left(-\frac{\pi}{3}\right)\right) = 2 - 2\sqrt{3} i$

$z_6 = 6\sqrt{2}\left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right) = -6 + 6i$

⑦ (a) $z = \frac{1-i}{1+i} = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i} = \frac{-2i}{2} = -i$

So $|z| = \sqrt{0^2 + (-1)^2} = 1$

Principal arg $\theta = \tan^{-1} \frac{-1}{0} = -\frac{\pi}{2}$

(b) $z = \frac{-1-7i}{4+3i} = \frac{-1-7i}{4+3i} \cdot \frac{4-3i}{4-3i} = \frac{1}{25}(-25-25i)$

$= -1-i$

$|z| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\arg \theta = \tan^{-1} \frac{-1}{-1} = \frac{\pi}{4}$, so Principal $\arg\theta = \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$

(c) $z = \frac{1+i}{2-i} = \frac{1+i}{2-i} \cdot \frac{2+i}{2+i} = \frac{1}{5}(1+3i)$

$\therefore |z| = \frac{1}{5}\sqrt{1^2+3^2} = \frac{\sqrt{10}}{5}$

& Principal arg $\theta = \tan^{-1}\left(\frac{3/5}{1/5}\right) = 1.25$ Radians

(d) $z = \frac{(3+i)^2}{1-i} = \frac{8+6i}{1-i} = \frac{8+6i}{1-i} \cdot \frac{1+i}{1+i} = \frac{2+14i}{2}$

$= 1+7i$

$\therefore |z| = \sqrt{1^2+7^2} = \sqrt{50} = 5\sqrt{2}$

& Principal arg $\theta = \tan^{-1} \frac{7}{1} = 1.43$

⑧ ⓐ given $z_1 = r_1 (\cos\theta_1 + i \cdot \sin\theta_1)$

& $z_2 = r_2 (\cos\theta_2 + i \cdot \sin\theta_2)$

Then $|z_1| = |r_1 (\cos\theta_1 + i \cdot \sin\theta_1)| = |r_1| \cdot |\cos\theta_1 + i \sin\theta_1|$

$= |r_1| \sqrt{\cos^2\theta_1 + \sin^2\theta_1}$

$= |r_1|$

& $|z_2| = |r_2 (\cos\theta_2 + i \cdot \sin\theta_2)| = |r_2| \cdot |\cos\theta_2 + i \cdot \sin\theta_2|$

$= |r_2| \cdot \sqrt{\cos^2\theta_2 + \sin^2\theta_2}$

$= |r_2|$

So $|z_1| \cdot |z_2| = |r_1| \cdot |r_2|$

Also $z_1 z_2 = r_1 \cdot r_2 (\cos\theta_1 + i \sin\theta_1) \cdot (\cos\theta_2 + i \cdot \sin\theta_2)$

$= r_1 \cdot r_2 (\cos\theta_1 \cos\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)$

$+ i^2 \sin\theta_1 \cdot \sin\theta_2)$

$= r_1 \cdot r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2$

$+ i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2))$

$= r_1 \cdot r_2 (\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2))$

Now $|z_1 . z_2| = |r_1 r_2 \ (\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2))|$

$= |r_1 . r_2| \ |\cos(\theta_1 + \theta_2) + i \cdot \sin(\theta_1 + \theta_2)|$

$= |r_1| . |r_2| \sqrt{\cos^2(\theta_1 + \theta_2) + \sin^2(\theta_1 + \theta_2)}$

$= |r_1| \ |r_2|$

$\therefore \quad |z_1 . z_2| = |z_1| . |z_2|$

Also $\arg(z_1) = \theta_1$ & $\arg(z_2) = \theta_2$

& $\arg(z_1 + z_2) = \theta_1 + \theta_2 = \arg(z_1) + \arg(z_2)$

(b) given $z_1 = 3\left(\cos\frac{\pi}{6} + i \cdot \sin\frac{\pi}{6}\right)$

& $z_2 = 2\left(\cos\frac{\pi}{4} + i \cdot \sin\frac{\pi}{4}\right)$

i) $|z_1 . z_2| = |z_1| . |z_2| = 3 \times 2 = 6$

ii) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$

iii) $|z_1^2| = |z_1|^2 = 3^2 = 9$

iv) $|z_2^2| = |z_2|^2 = 4$

v) $\arg(z_1^2) = \arg(z_1 . z_1) = \arg(z_1) + \arg(z_1) = 2\arg(z_1) = \frac{\pi}{3}$

vi) $\arg(z_2^2) = 2\arg(z_2) = \frac{\pi}{2}$ (see v) for Reasoning)

(9) (a) given $z_1 = r_1(\cos\theta_1 + i.\sin\theta_1)$
&
$z_2 = r_2(\cos\theta_2 + i.\sin\theta_2)$

Then $$\frac{|z_1|}{|z_2|} = \frac{|r_1(\cos\theta_1 + i.\sin\theta_1)|}{|r_2(\cos\theta_2 + i.\sin\theta_2)|}$$

$$= \frac{|r_1|}{|r_2|} \cdot \frac{|\cos\theta_1 + i.\sin\theta_1|}{|\cos\theta_2 + i.\sin\theta_2|}$$

$$= \frac{|r_1|}{|r_2|} \frac{\sqrt{\cos^2\theta_1 + \sin^2\theta_1}}{\sqrt{\cos^2\theta_2 + \sin^2\theta_2}}$$

$$= \frac{|r_1|}{|r_2|}$$

Also $$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = [r_1(\cos\theta_1 + i.\sin\theta_1)][r_2(\cos\theta_2 + i.\sin\theta_2)]^{-1}$$

$$= r_1 \cdot r_2^{-1}(\cos\theta_1 + i.\sin\theta_1)(\cos\theta_2 - i.\sin\theta_2)$$

$$= \frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) + \sin\theta_1 \cdot \sin\theta_2)$$

$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

So $$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1}{r_2}\right| \sqrt{\cos^2(\theta_1 - \theta_2) + \sin^2(\theta_1 - \theta_2)} = \left|\frac{r_1}{r_2}\right| = \frac{|r_1|}{|r_2|}$$

So $$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

Also $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2$ & $\arg(z_1) = \theta_1$; $\arg(z_2) = \theta_2$

$\therefore$ $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

(b) given $z_1 = 2\left(\cos\frac{2\pi}{3} + i.\sin 2\pi/3\right)$

& $z_2 = 6\left(\cos\left(-\frac{3\pi}{4}\right) + i.\sin\left(-\frac{3\pi}{4}\right)\right)$

Then i) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{1}{3}$

ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \frac{2\pi}{3} + \frac{3\pi}{4} = \frac{17\pi}{12}$

$= \frac{-7\pi}{12}$

iii) $\left|\frac{z_2}{z_1}\right| = \frac{|z_2|}{|z_1|} = 3$

iv) $\arg\left(\frac{z_2}{z_1}\right) = \arg(z_2) - \arg(z_1) = -\frac{3\pi}{4} - \frac{2\pi}{3} = \frac{-17\pi}{12}$

$= \frac{7\pi}{12}$

(10) given $z_1 = 4\left(\cos\frac{13\pi}{24} + i.\sin\frac{13\pi}{24}\right)$

& $z_2 = 2\left(\cos\frac{5\pi}{24} + i.\sin\frac{5\pi}{24}\right)$

Then $\frac{z_1}{z_2} = \frac{4}{2}\frac{(\cos 13\pi/24 + i.\sin 13\pi/24)}{(\cos 5\pi/24 + i.\sin 5\pi/24)}$

$= 2\left(\cos\frac{13\pi}{24} + i.\sin\frac{13\pi}{24}\right)\left(\cos\frac{5\pi}{24} + i.\sin\frac{5\pi}{24}\right)^{-1}$

$$\therefore \quad \frac{Z_1}{Z_2} = 2\left(\cos\frac{13\pi}{24} + i\sin\frac{13\pi}{24}\right)\left(\cos\frac{5\pi}{24} - i\cdot\sin\frac{5\pi}{24}\right)$$

$$= 2\left(\cos\frac{13\pi}{24}\cdot\cos\frac{5\pi}{24} + i\left(\sin\frac{13\pi}{24}\cos\frac{5\pi}{24} - \sin\frac{13\pi}{24}\cos\frac{5\pi}{24}\right) + \sin\frac{13\pi}{24}\cdot\sin\frac{5\pi}{24}\right)$$

$$= 2\left(\cos\left(\frac{13\pi}{24} - \frac{5\pi}{24}\right) + i\cdot\sin\left(\frac{13\pi}{24} - \frac{5\pi}{24}\right)\right)$$

$$= 2\left(\cos\left(\frac{8\pi}{24}\right) + i\sin\left(\frac{8\pi}{24}\right)\right) = 1 + \sqrt{3}\,i$$

For $Z_1 \cdot Z_2$ we have

$$Z_1 \cdot Z_2 = \left(4\left(\cos\frac{13\pi}{24} + i\sin\frac{13\pi}{24}\right)\right)\left(2\left(\cos\frac{5\pi}{24} + i\cdot\sin\frac{5\pi}{24}\right)\right)$$

$$= 8\left(\cos\left(\frac{13\pi}{24} + \frac{5\pi}{24}\right) + i\cdot\sin\left(\frac{13\pi}{24} + \frac{5\pi}{24}\right)\right)$$

$$= -4\sqrt{2} + 4\sqrt{2}\,i$$

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The claim is that if $z = r(\cos\theta + i\sin\theta)$,

$$[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)\,, \qquad (*)$$

is true for $n \in \mathbb{N}$.

Proof: Let $P(n)$ be $[r(\cos\theta + i.\sin\theta)]^n$

1. Base case: Let $n = 1$. Therefore the left hand side of (*) becomes

   $$[r(\cos\theta + i.\sin\theta)]^1 = r(\cos\theta + i\sin\theta)\,,$$

   and the right hand side of (*) becomes

   $$r^1(\cos(1\times\theta) + i\sin(1\times\theta)) = r(\cos\theta + i\sin\theta)\,.$$

   Since the left hand side equals the right hand side we have that $P(1)$ is true.

2. Inductive assumption – Let $n = k$: Let $P(k)$ be true for some positive integer k where $1 \leq k \leq n$. Then we have

   $$P(k)\colon [r(\cos\theta + i.\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)\,.$$

3. Let $n = k + 1$. We want to show that $P(k) \Longrightarrow P(k+1)$. Hence multiplying $P(k)$ by $r(\cos\theta + i\sin\theta)$ we obtain, by using our inductive assumption,

   $$\begin{aligned} &r^k(\cos k\theta + i\sin k\theta).r(\cos\theta + i\sin\theta) \\ &\qquad = r^{k+1}[(\cos k\theta\,.\cos\theta - \sin k\theta\,.\sin\theta) \\ &\qquad\qquad \times i(\cos k\theta\,.\sin\theta - \sin k\theta\,.\cos\theta)]\,. \end{aligned}$$

   Using standard trig identities we end up with

   $$\begin{aligned} &r^k(\cos k\theta + i\sin k\theta).r(\cos\theta + i\sin\theta) \\ &\qquad = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)\,, \end{aligned}$$

   which is $P(k+1)$, which is what we wanted to show.

Hence $P(k) \Longrightarrow P(k+1)$, and since $P(1)$ is true we have $P(n)$ is true for all $n \in \mathbb{N}$.

QED

(12) given $z = 2\left(\cos\frac{\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right)$

Then $z^6 = \left[2\left(\cos\frac{\pi}{3} + i\cdot\sin\frac{\pi}{3}\right)\right]^6$

$= 2^6\left(\cos\frac{6\pi}{3} + i\cdot\sin\frac{6\pi}{3}\right) = 64$

(13) given $z = 2\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right)$

then $z^6 = \left[2\left(\cos\frac{\pi}{4} + i\cdot\sin\frac{\pi}{4}\right)\right]^6$

$= 2^6\left(\cos\frac{6\pi}{4} + i\cdot\sin\frac{6\pi}{4}\right) = -64i$

(14) let $z^3 = 1$, $\therefore$ $z = 1^{1/3} = (\cos 2n\pi + i\cdot\sin 2n\pi)^{1/3}$

$= \cos\frac{2n\pi}{3} + i\cdot\sin\frac{2n\pi}{3}$

let $n = 0, 1, 2$. Hence

$n = 0$: $z_0 = \cos 0 + i\cdot\sin 0 = 1$

$n = 1$: $z_1 = \cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$n = 2$: $z_2 = \cos\frac{4\pi}{3} + i\cdot\sin\frac{4\pi}{3}$

$= \cos\left(-\frac{2\pi}{3}\right) + i\cdot\sin\left(-\frac{2\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

15 To Show That $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$

& $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$

Consider $\cos 3\theta + i \cdot \sin 3\theta = (\cos\theta + i\sin\theta)^3$

Now let $c = \cos\theta$ & $s = \sin\theta$.

$$\therefore \quad \cos 3\theta + i\sin 3\theta = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3$$

$$= c^3 - 3s^2 + i(3c^2 s - s^3)$$

Comparing Re & Im parts we have

$$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta \qquad (1)$$

and $$\sin 3\theta = 3\cos^2\theta\sin\theta - 3\sin^3\theta \qquad (2)$$

By (1) :
$$\cos 3\theta = \cos^3\theta - 3(1-\cos^2\theta)\cdot\cos\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

By (2) :
$$\sin 3\theta = 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

Hence $$\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta} = \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}.$$

Divide Through by $\cos^3\theta$: $$\tan 3\theta = \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta}$$

So $$\tan 3\theta = \frac{3\tan\theta(1+\tan^2\theta) - 4\tan^3\theta}{4 - 3(1+\tan^2\theta)} = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

(16) To show that $\sin 5\theta \equiv 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

use $\cos 5\theta + i.\sin 5\theta = (\cos\theta + i.\sin\theta)^5$

let $c \equiv \cos\theta$ & $s \equiv \sin\theta$, hence

$$\cos 5\theta + i.\sin 5\theta = c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$$

$$= c^5 - 10c^3s^2 + 5cs^4 + i(5c^4s - 10c^2s^3 + s^5)$$

Compare Im parts: $\sin 5\theta = 5\cos^4\theta.\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$$= 5(1-\sin^2\theta)^2.\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10(\sin^3\theta - \sin^5\theta) + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

(17) $\tan 6\theta = \dfrac{\sin 6\theta}{\cos 6\theta}$. So use

$$\cos 6\theta + i.\sin 6\theta = (\cos\theta + i.\sin\theta)^6.$$

let $c \equiv \cos\theta$ & $s = \sin\theta$, hence

$$\cos 6\theta + i.\sin 6\theta = c^6 + 6c^5(is) + 15c^4(is)^2 + 20c^3(is)^3 + 15c^2(is)^4 + 6c(is)^5 + (is)^6.$$

$\therefore \quad \cos 6\theta + i \sin 6\theta = C^6 - 15C^4S^2 + 15C^2S^4 - S^6$

$$+ i(6C^5S - 20C^3S^3 + 6CS^5)$$

comparing Re & Im parts, & forming $\tan 6\theta$, we have

$$\tan 6\theta = \frac{6C^5S - 20C^3S^3 + 6CS^5}{C^6 - 15C^4S^2 + 15C^2S^4 - S^6}$$

Now divide top & bottom by C^6:

$$\tan 6\theta = \frac{6\,S/C - 20\,S^3/C^3 + 6\,S^5/C^5}{1 - 15\,S^2/C^2 + 15\,S^4/C^4 - S^6/C^6}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

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Let $z = r(\cos\theta + i\sin\theta)$. Given that

$$[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)\,, \qquad (*)$$

is true for $n \in \mathbb{N}$, we want to show that it is also true for $n \in \mathbb{Z}$, i.e. when n is a negative integer.

<u>Proof:</u> The left hand side of (*) can be written as

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{1}{(\cos\theta + i\sin\theta)^m}.$$

where m is a positive integer. By (*) we have

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{1}{\cos m\theta + i\sin m\theta}.$$

Multiplying the right hand side by the conjugate of the denominator we have

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{1}{\cos m\theta + i\sin m\theta}.\frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta},$$

which ultimately simplfies to

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}(\cos(-m\theta) + i\sin(-m\theta))\,,$$

(since “$\cos^2$ $+ \sin^2$ " $= 1$) which is what we wanted to prove. Hence

$$[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

is true for all $n \in \mathbb{Z}$.